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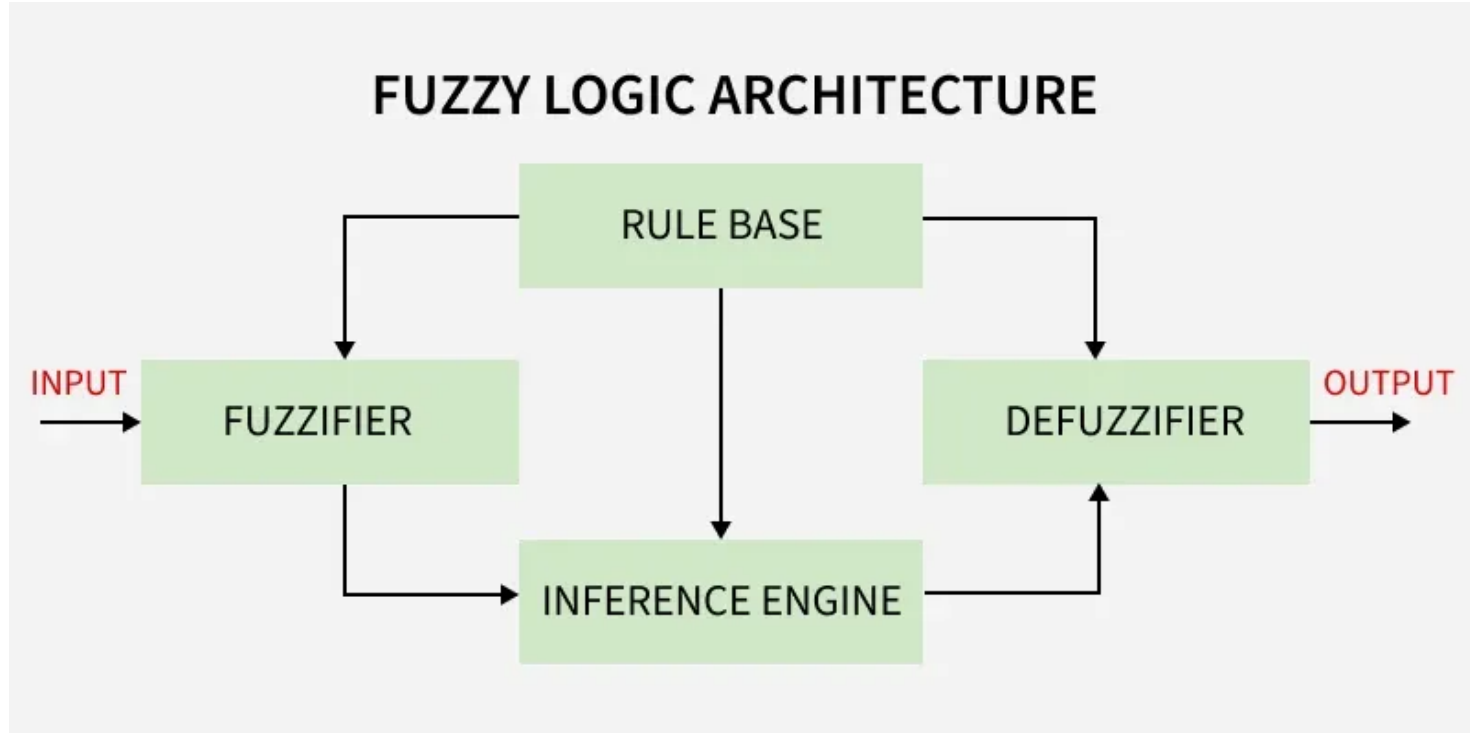
BASICS OF FUZZY LOGIC THEORY

- Fuzzy Logic system can work with any type of inputs whether it is imprecise, distorted or noisy input information.
- The construction of Fuzzy Logic Systems is easy and understandable.
- Fuzzy logic comes with mathematical concepts of set theory and the reasoning of that is quite simple.
- It provides a very efficient solution to complex problems in all fields of life as it resembles human reasoning and decision making.
- The algorithms can be described with little data, so little memory is required.

BASICS OF FUZZY LOGIC THEORY

- It is a technique to embody human-like thinkings into a control system.
- It may not be designed to give accurate reasoning but it is designed to give acceptable reasoning.
- It can emulate human deductive thinking, that is, the process people use to infer conclusions from what they know.
- Any uncertainties can be easily dealt with the help of fuzzy logic.

BASICS OF FUZZY LOGIC THEORY



BASICS OF FUZZY LOGIC THEORY

- **RULE BASE:** It contains the set of rules and the IF-THEN conditions provided by the experts to govern the decision making system, on the basis of linguistic information. Recent developments in fuzzy theory offer several effective methods for the design and tuning of fuzzy controllers. Most of these developments reduce the number of fuzzy rules.
- **FUZZIFICATION:** It is used to convert inputs i.e. crisp numbers into fuzzy sets. Crisp inputs are basically the exact inputs measured by sensors and passed into the control system for processing, such as temperature, pressure, rpm's, etc.

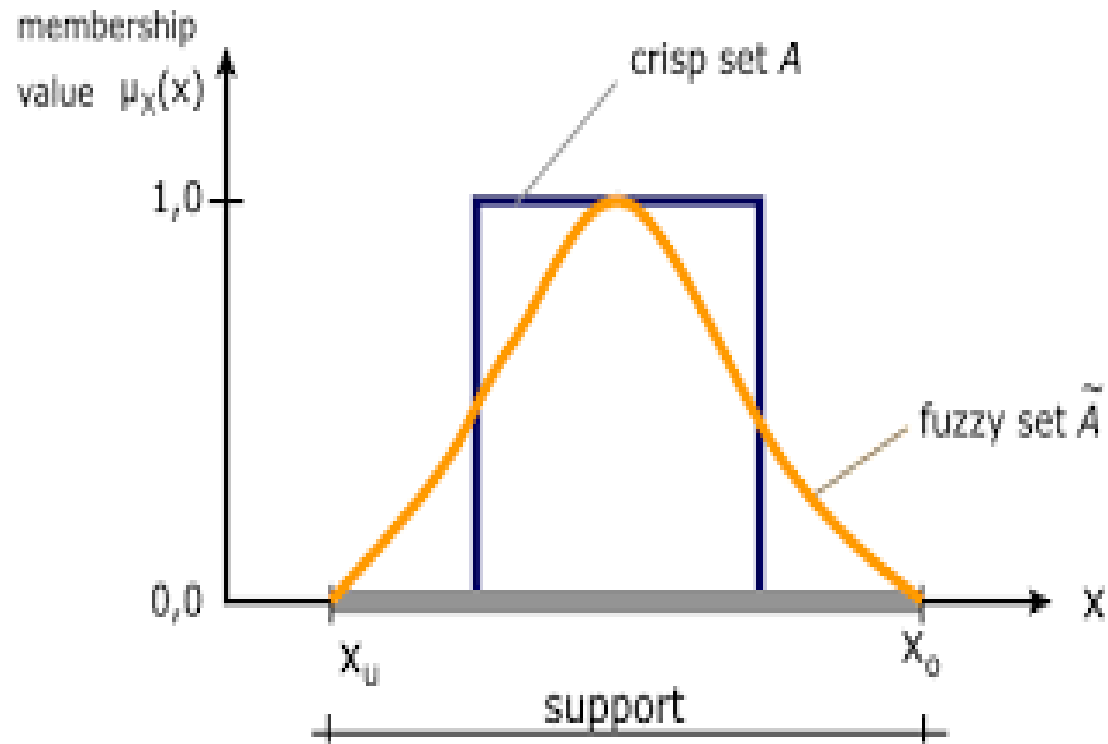
BASICS OF FUZZY LOGIC THEORY

- **INFERENCE ENGINE:** It determines the matching degree of the current fuzzy input with respect to each rule and decides which rules are to be fired according to the input field. Next, the fired rules are combined to form the control actions.
- **DEFUZZIFICATION:** It is used to convert the fuzzy sets obtained by inference engine into a crisp value. There are several defuzzification methods available and the best suited one is used with a specific expert system to reduce the error.

Crisp and Fuzzy Sets

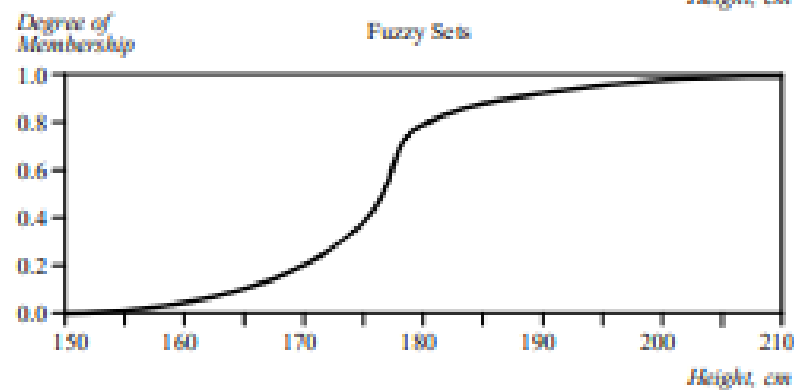
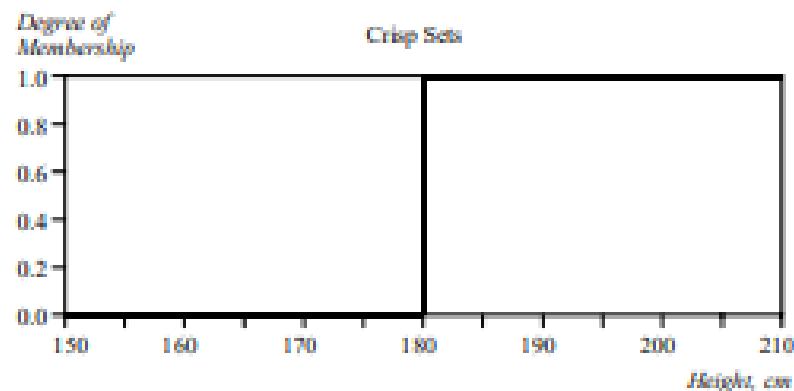
- Fuzzy set and crisp set are the part of the distinct set theories, where the fuzzy set implements infinite-valued logic while crisp set employs bi-valued logic.
- Previously, expert system principles were formulated premised on Boolean logic where crisp sets are used.
- But then scientists argued that human thinking does not always follow crisp “yes”/”no” logic, and it could be vague, qualitative, uncertain, imprecise or fuzzy in nature.
- This gave commencement to the development of the fuzzy set theory to imitate human thinking.

Crisp and Fuzzy Sets

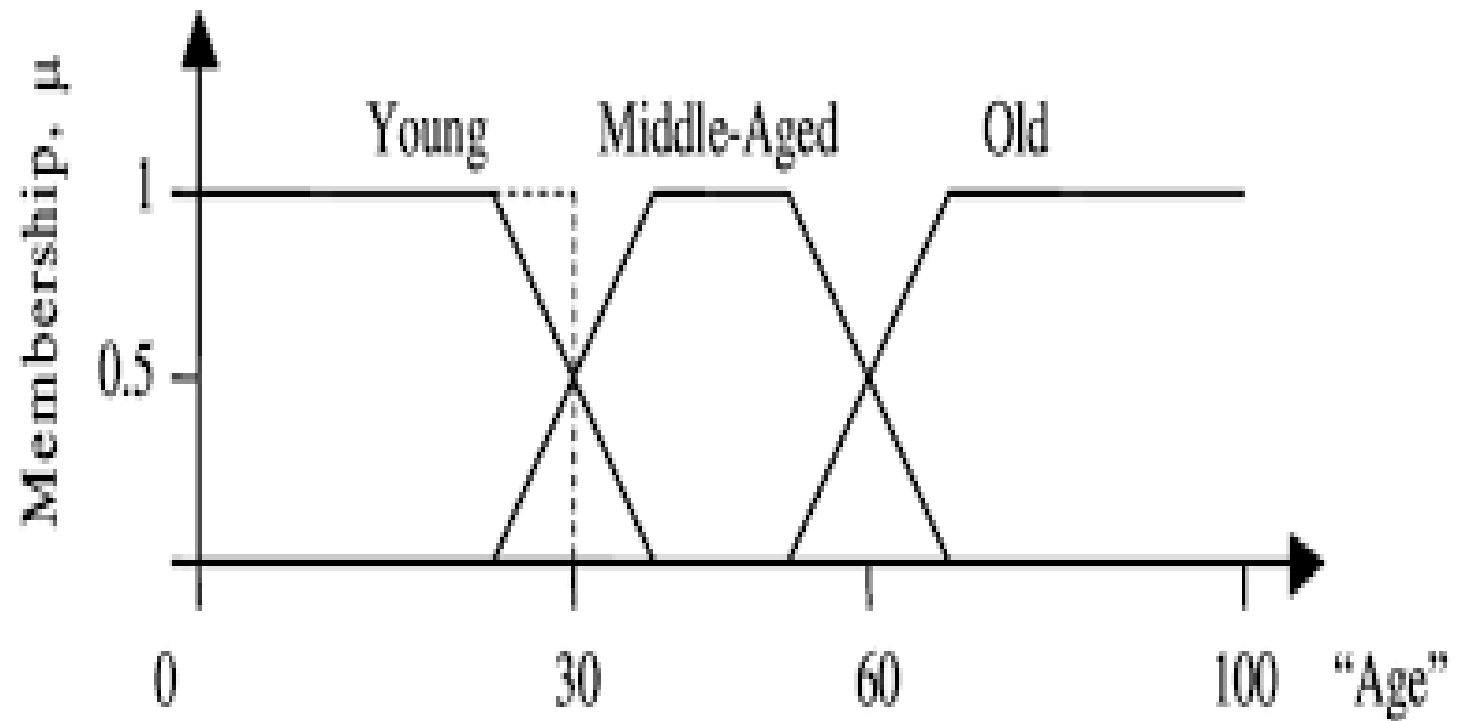


Crisp and Fuzzy Sets

Crisp and fuzzy sets of “*tall men*”



Crisp and Fuzzy Sets



Crisp/Classical Set Theory

Definition of A Crisp/Classical Set

- A **set** is an unordered collection of different elements. It can be written explicitly by listing its elements using the set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

Example

- A set of all positive integers.
- A set of all the planets in the solar system.
- A set of all the states in India.
- A set of all the lowercase letters of the alphabet.

Crisp/Classical Set Theory

Mathematical Representation of a Set

- A set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

Example

- Set of vowels in English alphabet,
 $A = \{a, e, i, o, u\}$
- Set of odd numbers less than 10,
 $B = \{1, 3, 5, 7, 9\}$

Crisp/Classical Set Theory

Set Builder Notation

- In this form, the set is defined by specifying a property that elements of the set have in common. The set is described as, $A = \{x:p(x)\}$

Example – The set $\{a,e,i,o,u\}$ is written as

- $A = \{x:x \text{ is a vowel in English alphabet}\}$

Example – The set $\{1,3,5,7,9\}$ is written as

- $B = \{x:1 \leq x < 10 \text{ and } (x\%2) \neq 0\}$

Crisp/Classical Set Theory

Member and Nonmember of a Set

- If an element x is a member of any set S , it is denoted by $x \in S$ and if an element y is not a member of set S , it is denoted by $y \notin S$.

Example

- If $S = \{1, 1.2, 1.7, 2\}$, $1 \in S$ but $1.5 \notin S$

Crisp/Classical Set Theory

Cardinality of a Set

- Cardinality of a set S , denoted by $|S|$, where $|S|$ is the number of elements of the set.
- The number is also referred as the cardinal number. If a set has an infinite number of elements, its cardinality is ∞ .

Example

- $|\{1,4,3,5\}| = 4$.
- $|\{1,2,3,4,5,\dots\}| = \infty$

Crisp/Classical Set Theory

Cardinality of a Set

- If there are two sets X and Y , $|X| = |Y|$ denotes two sets X and Y having same cardinality. It occurs when the number of elements in X is exactly equal to the number of elements in Y . In this case, there exists a bijective function 'f' from X to Y .
- $|X| \leq |Y|$ denotes that set X 's cardinality is less than or equal to set Y 's cardinality. It occurs when the number of elements in X is less than or equal to that of Y . Here, there exists an injective function 'f' from X to Y .

Crisp/Classical Set Theory

Cardinality of a Set

- $|X| < |Y|$ denotes that set X 's cardinality is less than set Y 's cardinality. It occurs when the number of elements in X is less than that of Y . Here, the function ' f ' from X to Y is injective function but not bijective.

Crisp/Classical Set Theory

Types of Sets

- Sets can be classified into many types; some of which are Universal, Finite, Infinite, Subset, Proper, Singleton Set, etc.

Universal Set

- It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U .

Example

- We may define U as the set of all animals on earth. In this case, a set of all mammals is a subset of U , a set of all fishes is a subset of U , a set of all insects is a subset of U , and so on.

Crisp/Classical Set Theory

Finite Set

- A set which contains a definite number of elements is called a Finite set.

Example

- $S = \{x | x \in \mathbb{N} \text{ and } 70 > x > 50\}$

Infinite Set

- A set which contains infinite number of elements is called an Infinite set.

Example

- $S = \{x | x \in \mathbb{N} \text{ and } x > 10\}$

Crisp/Classical Set Theory

Subset

- A set X is a subset of set Y (Written as $X \subseteq Y$) if every element of X is an element of set Y .

Example

- Let, $X = \{1,2,3,4,5,6\}$ and $Y = \{1,2\}$. Here set Y is a subset of set X as all the elements of set Y is in set X . Hence, we can write $Y \subset X$.

Example

- Let, $X = \{1,2,3\}$ and $Y = \{1,2,3\}$. Here set Y is a subset (not a proper subset) of set X as all the elements of set Y is in set X . Hence, we can write $Y \subseteq X$.

Crisp/Classical Set Theory

Proper Subset

- The term “proper subset” can be defined as “subset of but not equal to”. A Set X is a proper subset of set Y (Written as $X \subset Y$) if every element of X is an element of set Y and $|X| < |Y|$.

Example

- Let, $X = \{1,2,3,4,5,6\}$ and $Y = \{1,2\}$. Here set $Y \subset X$, since all elements in Y are contained in X too and X has at least one element which is more than set Y .

Crisp/Classical Set Theory

Empty Set or Null Set

- An empty set contains no elements. It is denoted by Φ . As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example

- $S = \{x|x \in \mathbb{N} \text{ and } 7 < x < 8\} = \Phi$

Singleton Set or Unit Set

- A Singleton set or Unit set contains only one element. A singleton set is denoted by $\{s\}$.

Example

- $S = \{x|x \in \mathbb{N}, 7 < x < 9\} = \{8\}$

Crisp/Classical Set Theory

Equal Set

- If two sets contain the same elements, they are said to be equal.

Example

- If $A = \{1,2,6\}$ and $B = \{6,1,2\}$, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

Equivalent Set

- If the cardinalities of two sets are same, they are called equivalent sets.

Example

- If $A = \{1,2,6\}$ and $B = \{16,17,22\}$, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. $|A| = |B| = 3$

Crisp/Classical Set Theory

Overlapping Set

- Two sets that have at least one common element are called overlapping sets.

Example

- Let, $A = \{1,2,6\}$ and $B = \{6,12,42\}$. There is a common element '6', hence these sets are overlapping sets.

Disjoint Set

- Two sets A and B are called disjoint sets if they do not have even one element in common. Therefore, disjoint sets have the following properties

$$n(A \cap B) = \phi$$

Example Let, $A = \{1,2,6\}$ and $B = \{7,9,14\}$, there is not a single common element, hence these sets are disjoint sets.

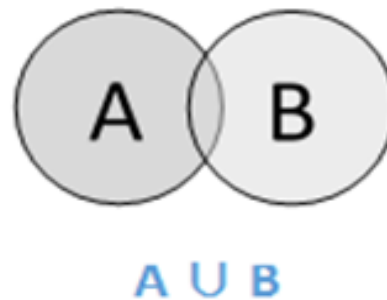
Crisp/Classical Set Theory

Operations on Crisp/Classical Sets

- Set Operations include Set Union, Set Intersection, Set Difference, Complement of Set, and Cartesian Product.

Union

- The union of sets A and B (denoted by $A \cup B$) is the set of elements which are in A , in B , or in both A and B . Hence, $A \cup B = \{x | x \in A \text{ OR } x \in B\}$.

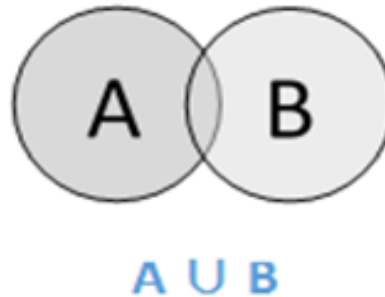


Crisp/Classical Set Theory

Union

Example

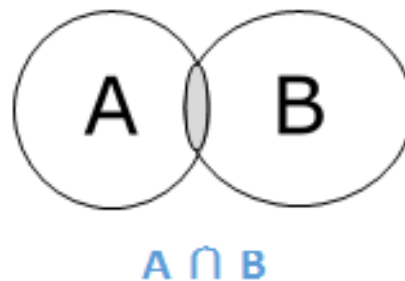
- If $A = \{10,11,12,13\}$ and $B = \{13,14,15\}$, then,
- $A \cup B = \{10,11,12,13,14,15\}$
- The common element occurs only once.



Crisp/Classical Set Theory

Intersection

- The intersection of sets A and B (denoted by $A \cap B$) is the set of elements which are in both A and B.
- Hence, $A \cap B = \{x | x \in A \text{ AND } x \in B\}$.

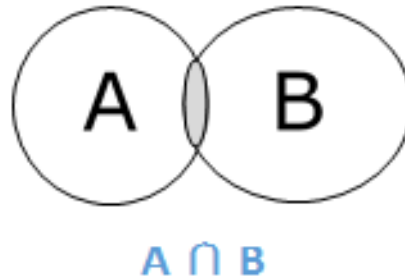


Crisp/Classical Set Theory

Intersection

Example

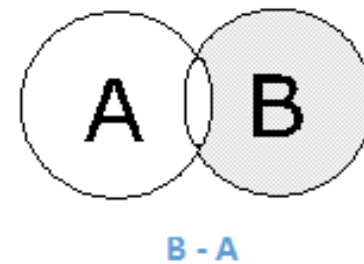
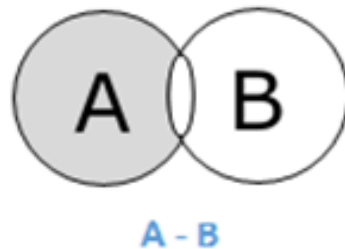
- If $A = \{10,11,12,13\}$ and $B = \{13,14,15\}$, then,
- $A \cap B = \{13\}$
- The common elements are taken only.



Crisp/Classical Set Theory

Difference/ Relative Complement

- The set difference of sets A and B (denoted by $A - B$) is the set of elements which are only in A but not in B .
- Hence, $A - B = \{x | x \in A \text{ AND } x \notin B\}$.

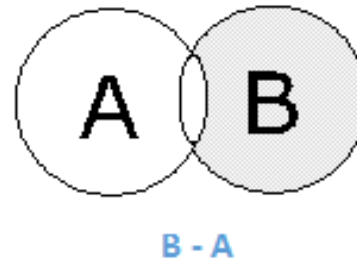
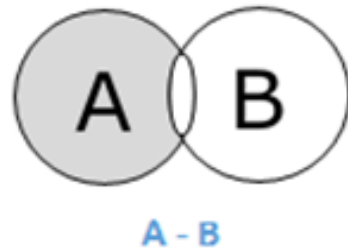


Crisp/Classical Set Theory

Difference/ Relative Complement

Example

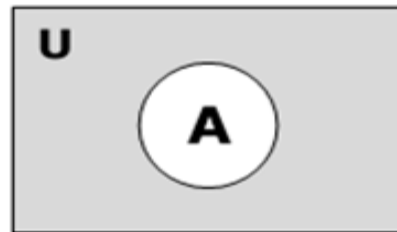
- If $A = \{10,11,12,13\}$ and $B = \{13,14,15\}$,
- $(A - B) = \{10,11,12\}$.
- $(B - A) = \{14,15\}$.
- Here, we can see $(A - B) \neq (B - A)$



Crisp/Classical Set Theory

Complement of a Set

- The complement of a set A (denoted by A') is the set of elements which are not in set A . Hence, $A' = \{x | x \notin A\}$.
- More specifically, $A' = (U - A)$ where U is a universal set which contains all objects.



Complement of set A

Crisp/Classical Set Theory

Complement of a Set

Example

- If $A = \{x | x \text{ belongs to set of odd integers}\}$ then $A' = \{y | y \text{ does not belong to set of odd integers}\}.$

Crisp/Classical Set Theory

Cartesian Product / Cross Product

- The Cartesian product of n number of sets $A_1, A_2, \dots, A_n$ denoted as $A_1 \times A_2 \times \dots \times A_n$ can be defined as all possible ordered pairs $(x_1, x_2, \dots, x_n)$ where $x_1 \in A_1, x_2 \in A_2, \dots, x_n \in A_n$

Example – If we take two sets $A = \{a, b\}$ and $B = \{1, 2\}$,

- The Cartesian product of A and B is written as, $A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$
- And the Cartesian product of B and A is written as, $B \times A = \{(1, a), (1, b), (2, a), (2, b)\}$

Crisp/Classical Set Theory

Properties of Classical Sets

- Properties on sets play an important role for obtaining the solution. Following are the different properties of classical sets.

Commutative Property

- Having two sets A and B, this property states;

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Crisp/Classical Set Theory

Associative Property

- Having three sets A, B and C, this property states;

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Crisp/Classical Set Theory

Distributive Property

- Having three sets A, B and C, this property states;

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Crisp/Classical Set Theory

Idempotency Property

- For any set A, this property states;

$$A \cup A = A$$

$$A \cap A = A$$

Crisp/Classical Set Theory

Identity Property

- For set A and universal set X , this property states;

$$A \cup \varnothing = A$$

$$A \cap X = A$$

$$A \cap \varnothing = \varnothing$$

$$A \cup X = X$$

Crisp/Classical Set Theory

Transitive Property

- Having three sets A, B and C, the property states;

$$\text{If } A \subseteq B \subseteq C, \text{ then } A \subseteq C$$

Involution Property

- For any set A, this property states –

$$\overline{\overline{A}} = A$$

Crisp/Classical Set Theory

De Morgan's Law

- It is a very important law and supports in proving tautologies and contradiction. This law states;

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Crisp/Classical Set Theory

HOME TASK 1

Let U be the Universal Set defined as;

$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$.

Let A and B are two sets defined as;

$A = \{2, 5, 7, 8, 10, 13, 15, 16, 18, 19, 20\}$

$B = \{1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 14, 16, 17, 19\}$

Crisp/Classical Set Theory

HOME TASK 1

Then Find out the following:

1. $A \cup B$
2. $A \cap B$
3. $(A - B)$
4. $(B - A)$
5. A'
6. B'

Crisp/Classical Set Theory

HOME TASK 2

Let A and B are two crisp sets defined as;

$$A = \{R, S\}.$$

$$B = \{7, 10, 18\}$$

Then Find out the Cartesian Products as follows:

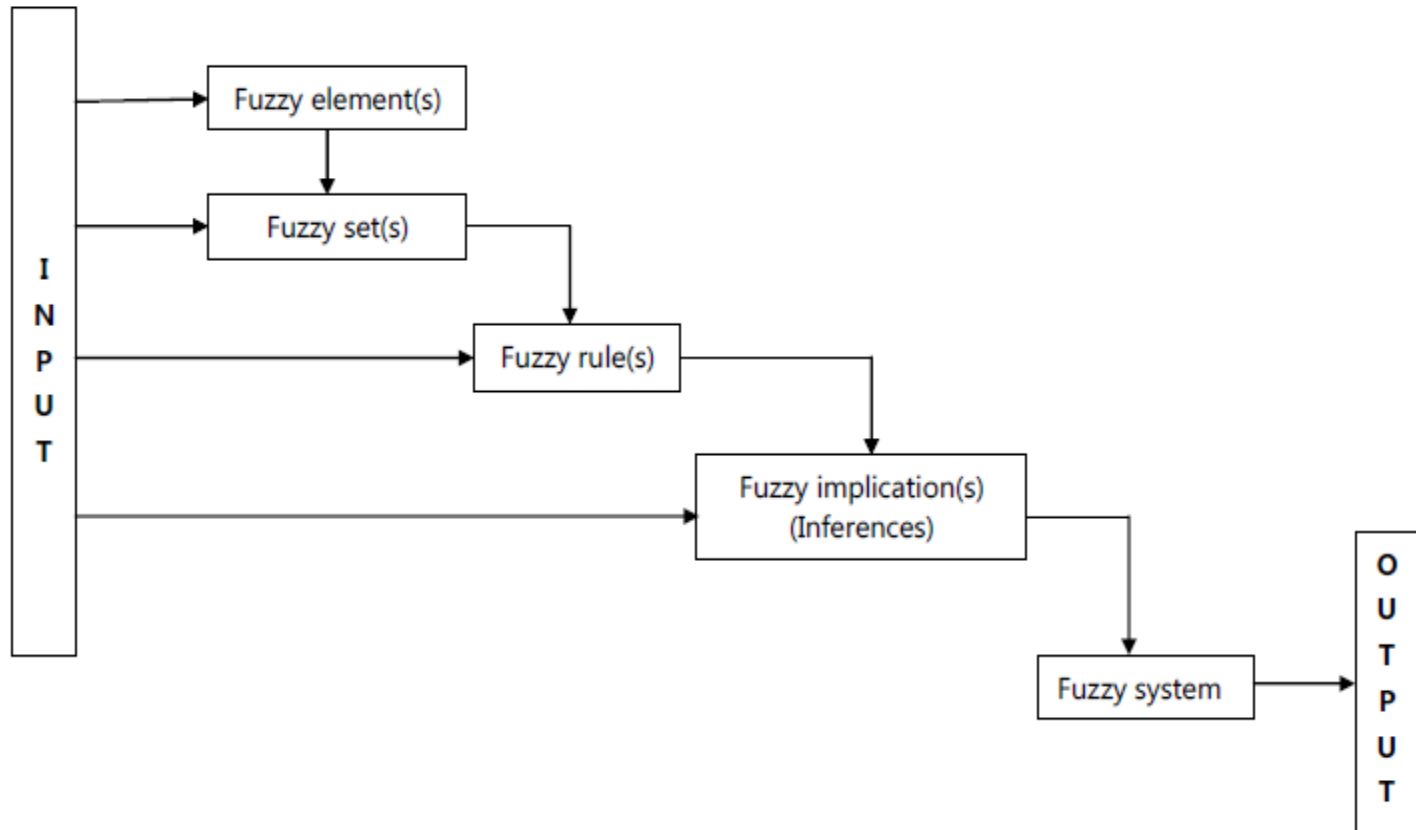
1. $A \times B$
2. $B \times A$

Crisp/Classical Set Theory

HOME TASK 3

- Give Examples one from each for an Empty Set, Unit Set, Equal Set, Equivalent Set, Overlapping Set, Disjoint Set.

Concept of Fuzzy System



Fuzzy Sets

- Fuzzy sets can be considered as an extension and gross oversimplification of classical sets.
- It can be best understood in the context of set membership. Basically it allows partial membership which means that it contain elements that have varying degrees of membership in the set.

Classical Set

Classical set

1. Classical set is a collection of **distinct** objects. For example, a set of students passing grades.
2. Each individual entity in a set is called a **member** or an **element** of the set.
3. The classical set is defined in such a way that the universe of discourse is spitted into two groups **members** and **non-members**. Hence, In case classical sets, **no partial membership exists**.
4. Let A is a given set. The membership function can be use to define a set A is given by:

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Fuzzy Set

Fuzzy set:

1. **Fuzzy set** is a set having **degrees of membership** between 1 and 0. Fuzzy sets are represented with tilde character($\sim$). For example, Number of cars following traffic signals at a particular time out of all cars present will have membership value between $[0,1]$.
2. Partial membership exists when member of one fuzzy set can also be a part of other fuzzy sets in the same universe.
3. The degree of membership or truth is not same as probability, fuzzy truth represents membership in vaguely defined sets.
4. A fuzzy set $A_{\sim}$ in the universe of discourse, U , can be defined as a set of ordered pairs and it is given by

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$$

Difference between Classical Set and Fuzzy Set

Crisp Set	Fuzzy Set
1. $S = \{ s \mid s \in X \}$	1. $F = (s, \mu) \mid s \in X \text{ and } \mu(s) \text{ is the degree of } s.$
2. It is a collection of elements.	2. It is collection of ordered pairs.
3. Inclusion of an element $s \in X$ into S is crisp, that is, has strict boundary yes or no .	3. Inclusion of an element $s \in X$ into F is fuzzy, that is, if present, then with a degree of membership .

Difference between Classical Set and Fuzzy Set

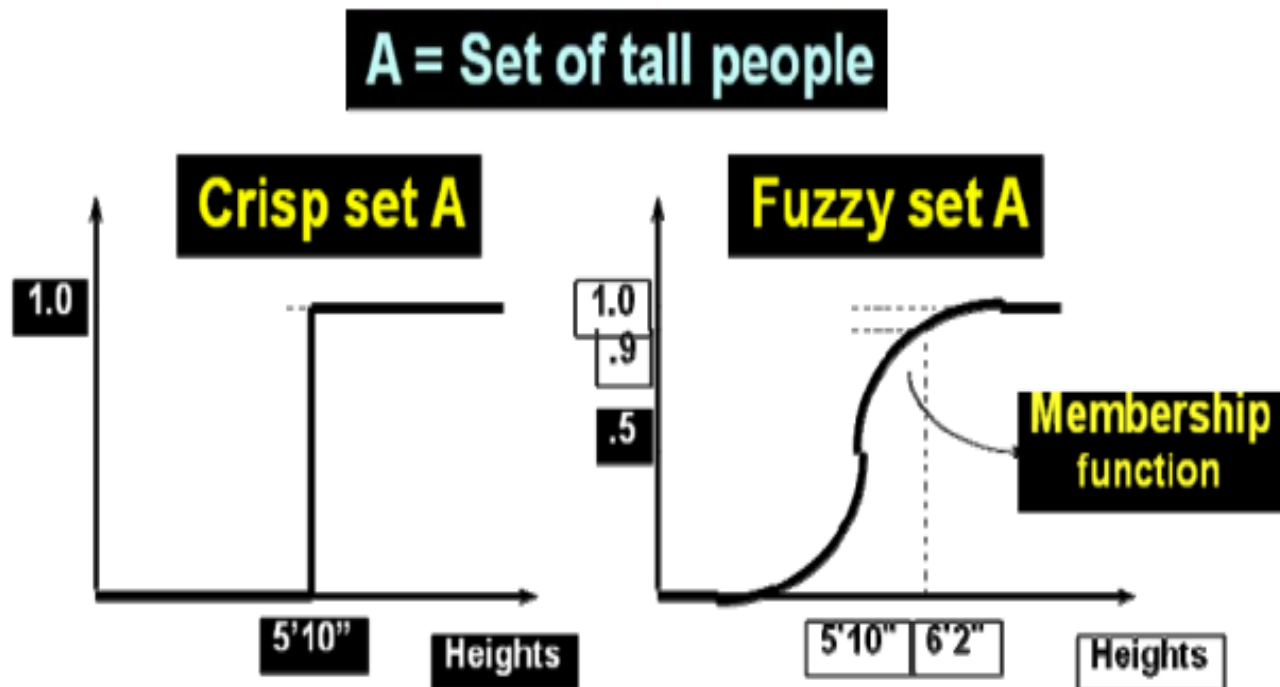
BASIS FOR COMPARISON	FUZZY SET	CRISP SET
Basic	Prescribed by vague or ambiguous properties.	Defined by precise and certain characteristics.
Property	Elements are allowed to be partially included in the set.	Element is either the member of a set or not.
Applications	Used in fuzzy controllers	Digital design
Logic	Infinite-valued	bi-valued

Difference between Classical Set and Fuzzy Set

Crisp vs. Fuzzy

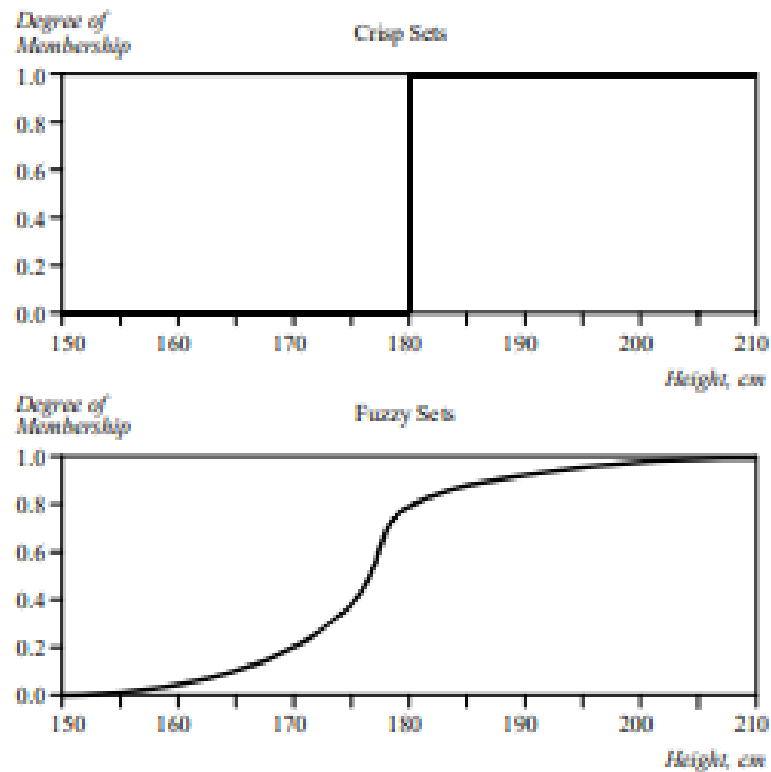
Crisp	Fuzzy
It has strict boundary T or F	Fuzzy boundary with a degree of membership
Some crisp time set can be fuzzy	It can't be crisp
True/False {0,1}	Membership values on [0,1]
In Crisp logic law of Excluded Middle and Non- Contradiction may or may not hold	In the fuzzy logic law of Excluded Middle and Non- Contradiction hold

Difference between Classical Set and Fuzzy Set

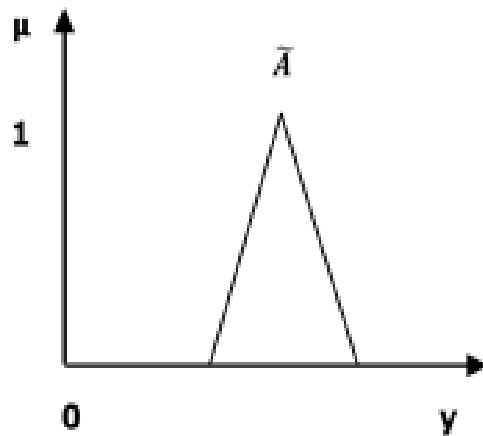


Crisp and Fuzzy Sets

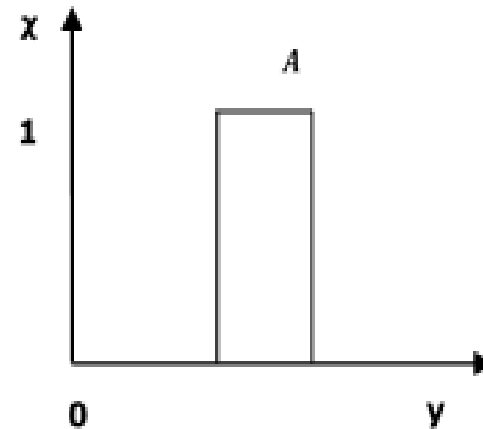
Crisp and fuzzy sets of “*tall men*”



Difference between Classical Set and Fuzzy Set



Membership Function of Fuzzy set $\tilde{A}$



Membership Function of classical set A

Fuzzy Set

Definition 1: Membership function (and Fuzzy set)

If X is a universe of discourse and $x \in X$, then a fuzzy set A in X is defined as a set of ordered pairs, that is

$A = \{(x, \mu_A(x)) | x \in X\}$ where $\mu_A(x)$ is called the **membership function** for the fuzzy set A .

Note:

$\mu_A(x)$ map each element of X onto a membership grade (or membership value) between 0 and 1 (both inclusive).

Fuzzy Set

- Let us consider fuzzy set $\tilde{A}$, $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$ where $\mu_{\tilde{A}}(x)$ is called the membership function for the fuzzy set $\tilde{A}$. X is referred to as the universe of discourse.
- The membership function associates each element $x \in X$ with a value in the interval $[0, 1]$.
- In fuzzy sets, each elements is mapped to $[0,1]$ by membership function. That is, $\mu_{\tilde{A}} : X \rightarrow [0, 1]$, where $[0,1]$ means real numbers between 0 and 1 (including 0,1).
- Consequently, fuzzy set is with 'vague boundary set' comparing with crisp set.

Fuzzy Set

- Here, $\mu_{\tilde{A}}(x)$ is the “membership function”. Value of this function is between 0 and 1.
- This value represents the “degree of membership” (membership value) of element x in set $\tilde{A}$.
- The members of a fuzzy set are members to some degree, known as a membership grade or degree of membership.
- The membership grade is the degree of belonging to the fuzzy set. The larger the number (in $[0, 1]$) the more the degree of belonging. (N.B. This is not a probability).
- The translation from x to $\mu_{\tilde{A}}(x)$ is known as Fuzzification.

Fuzzy Set

- In the fuzzy theory, fuzzy set $\tilde{A}$ of universe X is defined by function $\mu_{\tilde{A}}(x)$ called the membership function of set $\tilde{A}$.
- We already discussed this point. $\mu_{\tilde{A}}(x): X \rightarrow [0, 1]$, where $\mu_{\tilde{A}}(x) = 1$ if x is totally in $\tilde{A}$; $\mu_{\tilde{A}}(x) = 0$ if x is not in $\tilde{A}$; $0 < \mu_{\tilde{A}}(x) < 1$ if x is partly in $\tilde{A}$.

Fuzzy Set

Example:

X = All cities in India

A = City of comfort

$A = \{(\text{New Delhi}, 0.7), (\text{Bangalore}, 0.9), (\text{Chennai}, 0.8), (\text{Hyderabad}, 0.6), (\text{Kolkata}, 0.3), (\text{Kharagpur}, 0)\}$

Fuzzy Set

- **Formal definition:**

A fuzzy set A in X is expressed as a set of ordered pairs:

$$A = \{(x, \mu_A(x)) | x \in X\}$$

Fuzzy set

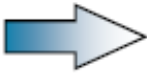
Membership
function
(MF)

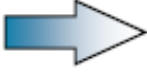
X : Universe or
universe of discourse

A fuzzy set is totally characterized by a membership function (MF).

Fuzzy Set(Alternative Notation)

- A fuzzy set A can be alternatively denoted as follows:

X is discrete  $A = \sum_{x_i \in X} \mu_A(x_i) / x_i$

X is continuous  $A = \int_X \mu_A(x) / x$

Note that Σ and integral signs stand for the union of membership grades; “/” stands for a marker and does not imply division.

Fuzzy Set(Alternative Notation)

- The fuzzy set $\tilde{A}$ can be alternatively denoted as follows: If X is discrete then $\tilde{A} = \sum \mu_{\tilde{A}}(x_i) / x_i$
- If X is continuous then $\tilde{A} = \int \mu_{\tilde{A}}(x) / x$

Fuzzy Set(Alternative Notation)

- **The notation for fuzzy sets:** for the member, x , of a discrete set with membership μ , we use the notation μ/x . In other words, x is a member of the set to degree μ .
- **Discrete sets** are defined as:

$$A = \mu_1/x_1 + \mu_2/x_2 + \dots + \mu_n/x_n$$

- or (in a more compact form)

$$A = \sum_{i=1 \dots n} \mu_i / x_i$$

$x_1, x_2, \dots, x_n$: members of the set A

$\mu_1, \mu_2, \dots, \mu_n$: $x_1, x_2, \dots, x_n$'s degree of membership.

Representation of a Fuzzy Set

Mathematical Concept

A fuzzy set $\tilde{A}$ in the universe of information U can be defined as a set of ordered pairs and it can be represented mathematically as –

$$\tilde{A} = \{ (y, \mu_{\tilde{A}}(y)) \mid y \in U \}$$

Here $\mu_{\tilde{A}}(y)$ = degree of membership of y in $\tilde{A}$, assumes values in the range from 0 to 1, i.e., $\mu_{\tilde{A}}(y) \in [0, 1]$.

Representation of a Fuzzy Set

- Let us now consider two cases of universe of information and understand how a fuzzy set can be represented.

Case 1

When universe of information U is discrete and finite –

$$\begin{aligned}\tilde{A} &= \left\{ \frac{\mu_{\tilde{A}}(y_1)}{y_1} + \frac{\mu_{\tilde{A}}(y_2)}{y_2} + \frac{\mu_{\tilde{A}}(y_3)}{y_3} + \dots \right\} \\ &= \left\{ \sum_{i=1}^n \frac{\mu_{\tilde{A}}(y_i)}{y_i} \right\}\end{aligned}$$

Representation of a Fuzzy Set

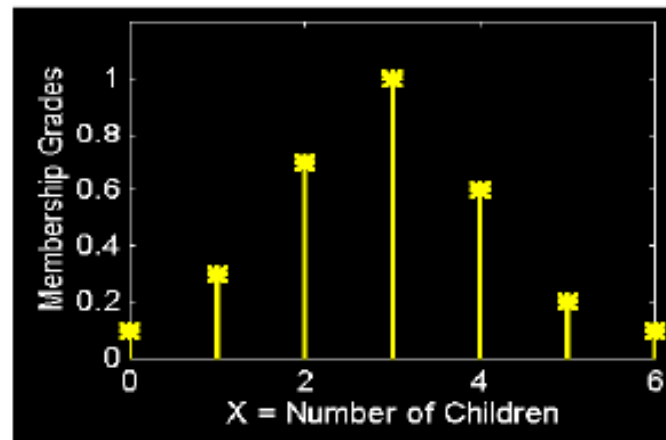
Case 2

When universe of information U is continuous and infinite –

$$\tilde{A} = \left\{ \int \frac{\mu_{\tilde{A}}(y)}{y} \right\}$$

Fuzzy Sets with Discrete Universes

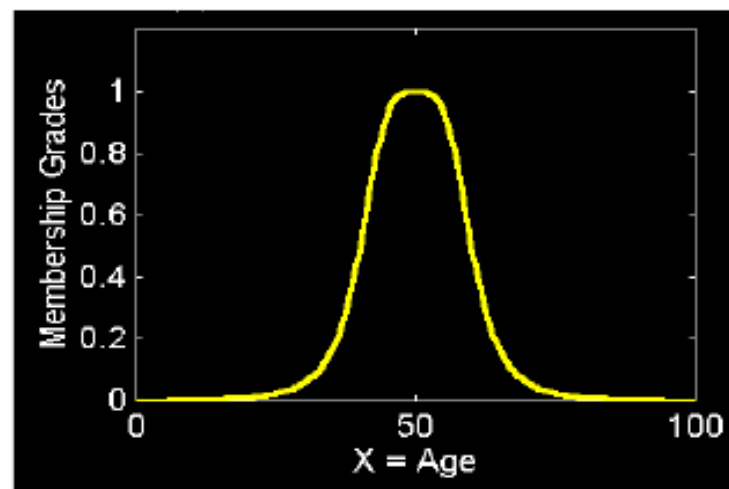
- Fuzzy set **C** = “desirable city to live in”
 $X = \{\text{Tehran, Tabriz, Rasht}\}$ (**discrete and non-ordered**)
 $C = \{(\text{Tehran}, 0.1), (\text{Tabriz}, 0.8), (\text{Rasht}, 0.9)\}$
- Fuzzy set **A** = “sensible number of children in a family”
 $X = \{0, 1, 2, 3, 4, 5, 6\}$ (**discrete ordered universe**)
 $A = \{(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)\}$



Fuzzy Sets with Continuous Universes

- Fuzzy set **B** = “about 50 years old”
X = Set of positive real numbers (**continuous**)
 $B = \{(x, \mu_B(x)) \mid x \text{ in } X\}$

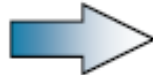
$$\mu_B(x) = \frac{1}{1 + \left(\frac{x - 50}{10}\right)^2}$$



Fuzzy Set(Alternative Notation)

- Examples:

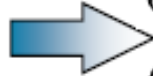
A is discrete



$$A = \{(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)\}.$$

$$A = 0.1/0 + 0.3/1 + 0.7/2 + 1.0/3 + 0.7/4 + 0.3/5 + 0.1/6,$$

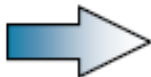
C is discrete



$$C = \{(\text{San Francisco}, 0.9), (\text{Boston}, 0.8), (\text{Los Angeles}, 0.6)\}.$$

$$C = 0.9/\text{San Francisco} + 0.8/\text{Boston} + 0.6/\text{Los Angeles},$$

B is continuous



$$B = \text{"about 50 years old"} \quad \mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^4}.$$

$$B = \{(x, \mu_B(x)) | x \in X\},$$

$$B = \int_{R^+} \frac{1}{1 + \left(\frac{x-50}{10}\right)^4} / x,$$

Fuzzy Set Theory

HOME TASK 4

Give Examples of Fuzzy Sets with;

- Discrete non-ordered Universe.
- Discrete ordered Universe.
- Continuous Universe.

Fuzzy Set Theory

HOME TASK 5

- Represent the Alternative Notation of the following Fuzzy Set.

1. $A = \{(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)\}$

2. $C = \{(\text{Delhi}, 0.5), (\text{Mumbai}, 0.7), (\text{Kolkata}, 0.6), (\text{Chennai}, 0.2), (\text{Bhubaneswar}, 0.9)\}$

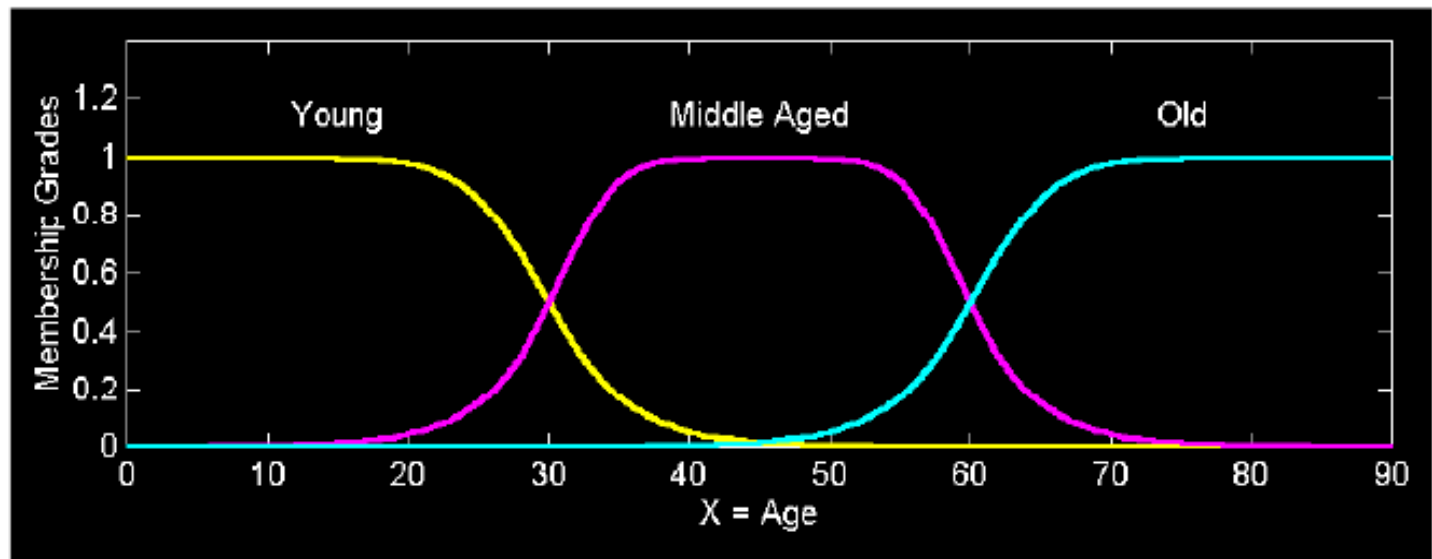
3. $\mu_A(x) = \frac{1}{1+x^2}$

Basic Concepts of Fuzzy Sets (Some Definitions)

- Linguistic variable
- Linguistic value
- Support
- Core
- Normality
- Crossover points
- Fuzzy singleton
- α -cut, strong α -cut
- Convexity
- Fuzzy numbers
- Bandwidth
- Symmetricity
- Open left or right, closed

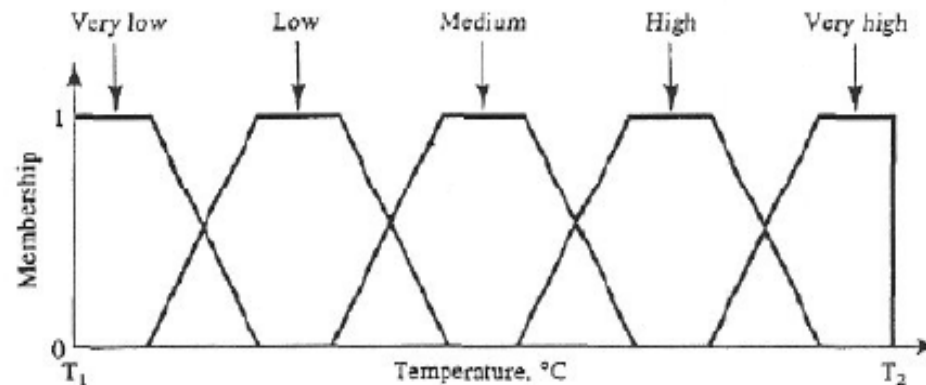
Fuzzy Partition

- Fuzzy partitions formed by the linguistic values “young”, “middle aged”, and “old”:



Fuzzy Partition

- Fuzzy partitions formed by the linguistic values “very low”, “low”, “medium”, “high”, and “very high”:

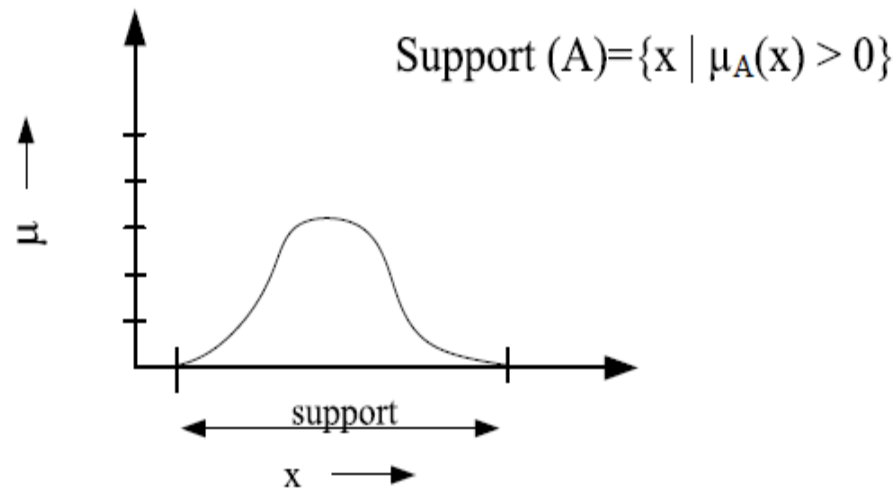


Linguistic terms

- Linguistic variable
 - In the first example “age” and in the second one “temperature” are linguistic variables.
- Linguistic value
 - In the first example “young” and in the second one “high” are linguistic values.

Support of a Fuzzy Set

Support: The support of a fuzzy set A is the set of all points $x \in X$ such that $\mu_A(x) > 0$



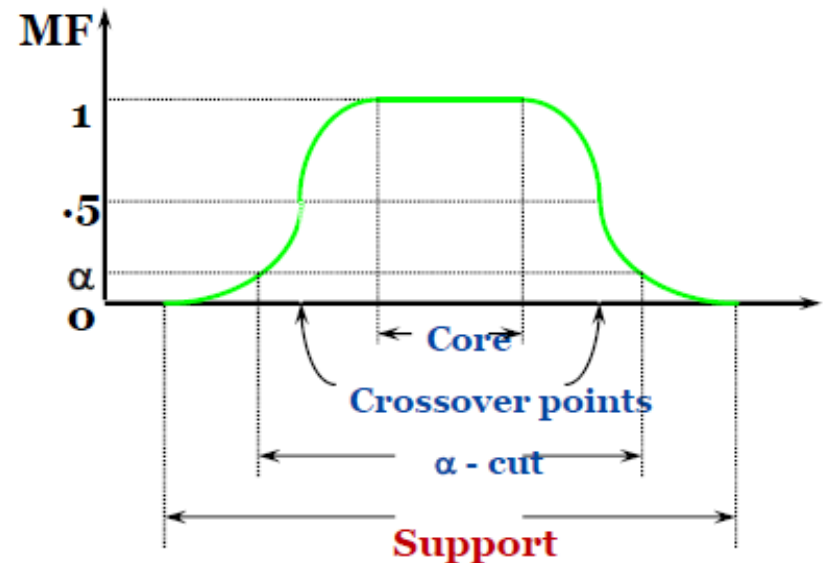
Support of a Fuzzy Set

- **Support**
- The Support of a fuzzy set **A** is the set of all points **x** in **X** such that:

$$\mu_A(x) > 0$$

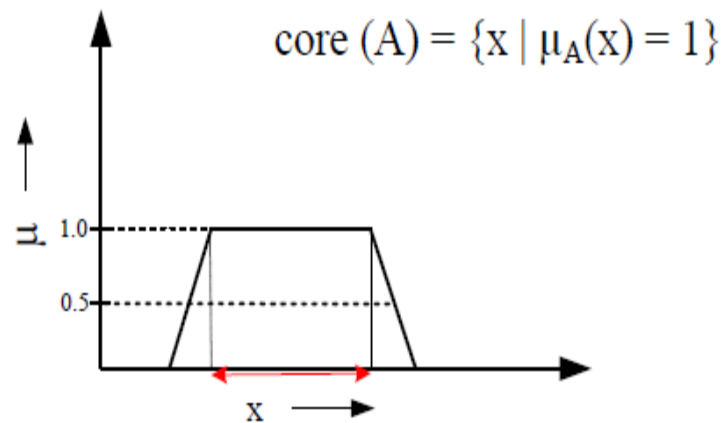
- In other words:

$$\text{support}(A) = \{x | \mu_A(x) > 0\}.$$



Core of a Fuzzy Set

Core: The core of a fuzzy set A is the set of all points x in X such that $\mu_A(x) = 1$

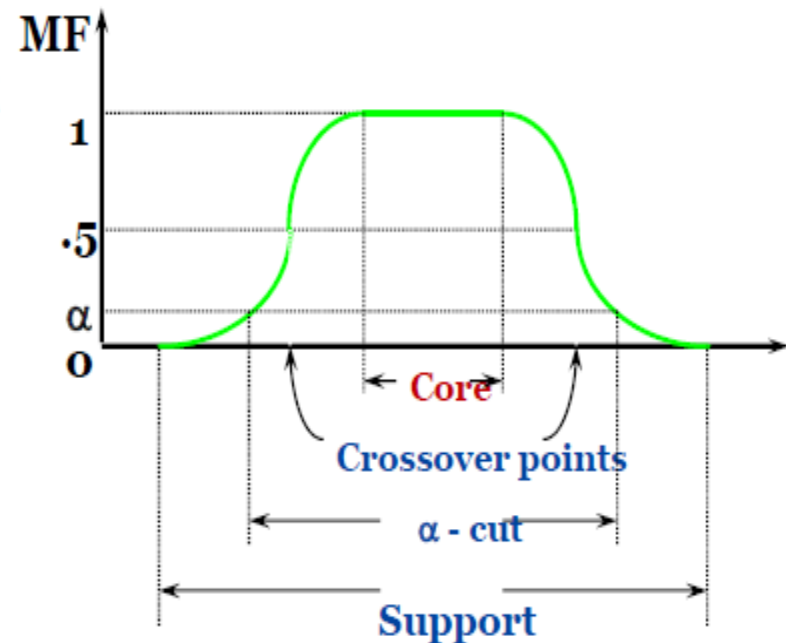


Core of a Fuzzy Set

- **Core**
- The Core of a fuzzy set **A** is the set of all points **x** in **X** such that:

$$\mu_A(x) = 1$$

- In other words:



Empty Fuzzy Set

Empty Fuzzy Set



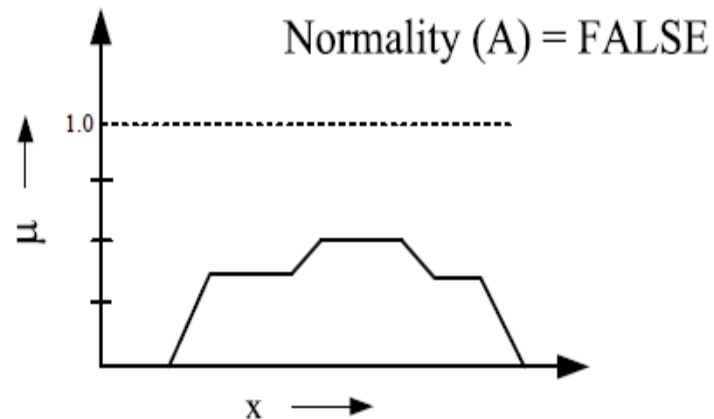
- A fuzzy set ($A=\emptyset$) is **empty** if its membership function is zero everywhere in its universe of discourse

$$A = \emptyset \quad \text{if} \quad \mu_A(x) = 0, \forall x \in X$$

An empty fuzzy set has an empty support

Normality of a Fuzzy Set

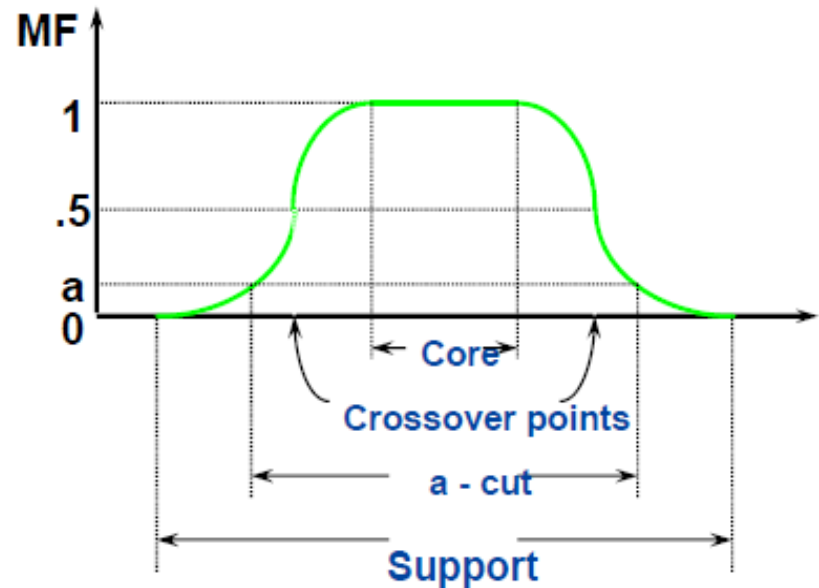
Normality : A fuzzy set A is a normal if its core is non-empty. In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$.



Normality of a Fuzzy Set

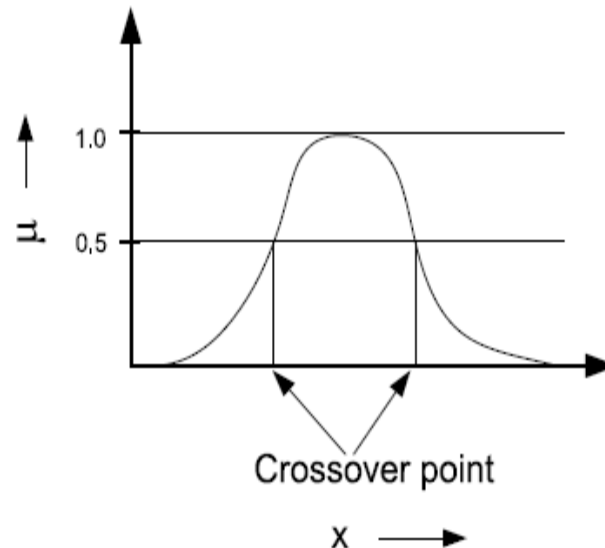
- **Normality:**
- Fuzzy set A is normal if its core is non-empty.
- In other words:
- We can always find a point x in X for which:

$$\mu_A(x) = 1.$$



Crossover point of a Fuzzy Set

Crossover point : A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$. That is
 $\text{Crossover}(A) = \{x | \mu_A(x) = 0.5\}$.



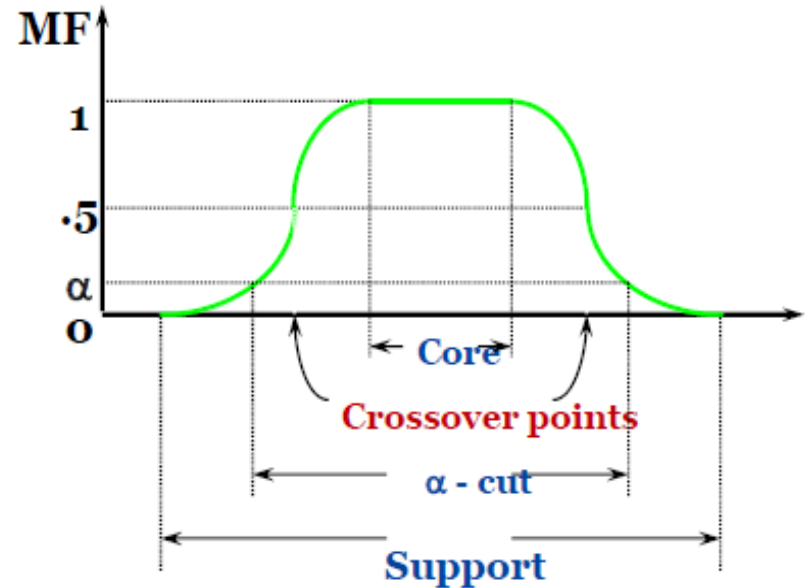
Crossover point of a Fuzzy Set

- **Crossover Point**
- The Crossover point of a fuzzy set **A** is a point **x** in **X** such that:

$$\mu_A(x) = 0.5$$

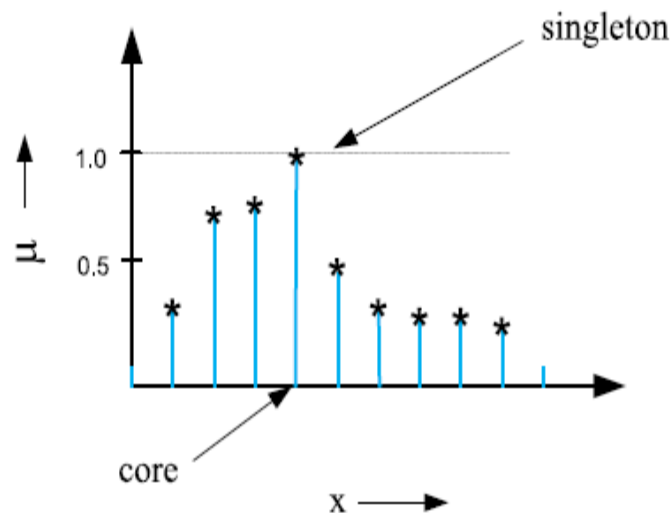
- In other words:

$$\text{crossover}(A) = \{x | \mu_A(x) = 0.5\}.$$



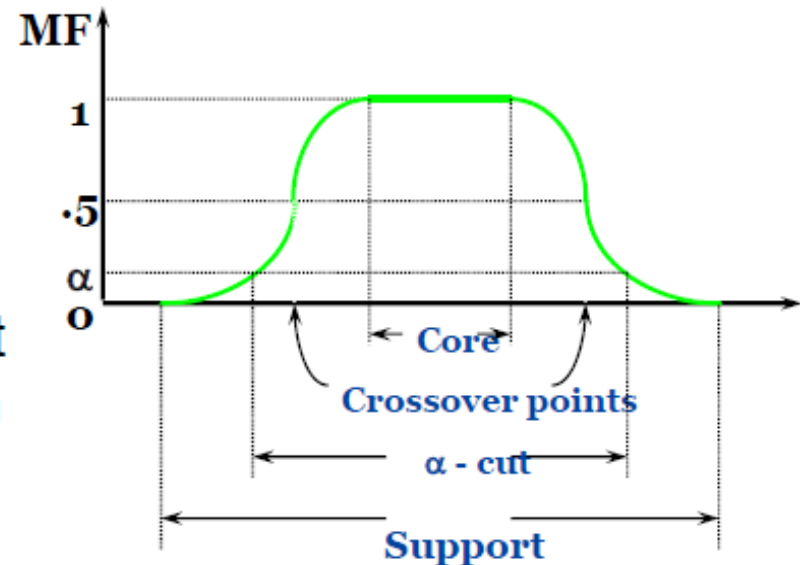
Fuzzy Singleton

Fuzzy Singleton : A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = |\{x \mid \mu_A(x) = 1\}| = 1$. Following fuzzy set is not a fuzzy singleton.

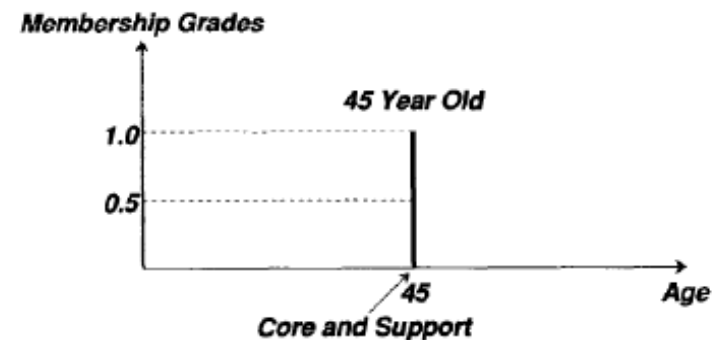


Fuzzy Singleton

- **Fuzzy Singleton:**
- A fuzzy set whose support is a single point in X is a fuzzy singleton if: $\mu_A(x) = 1$.



- Example:
- A fuzzy singleton
- “45 years old”



Height of a Fuzzy Set

- The **height** of a fuzzy set A :
 - The **height** of a fuzzy set A is the largest membership grade obtained by any element in that set.

$$h(A) = \sup_{x \in X} A(x)$$

- A fuzzy set A is called **normal** when $h(A) = 1$.
- It is called **subnormal** when $h(A) < 1$.
- The height of A may also be viewed as the supremum of α for which $A_\alpha \neq \Phi$.

Height of a Fuzzy Set

Height of a fuzzy set

- The **height** of a fuzzy set is the **highest membership** value of its membership function

$$H_A = \max_{x \in X} \{\mu_A(x)\}$$

- A fuzzy set is defined as normal when $H_A=1$ and subnormal when it is not.

α -cut and strong α -cut

α -cut and strong α -cut :

The α -cut of a fuzzy set A is a crisp set defined by

$$A_{\alpha} = \{x \mid \mu_A(x) \geq \alpha \}$$

Strong α -cut is defined similarly :

$$A_{\alpha}' = \{x \mid \mu_A(x) > \alpha \}$$

Note : $\text{Support}(A) = A_0$ and $\text{Core}(A) = A_1$.

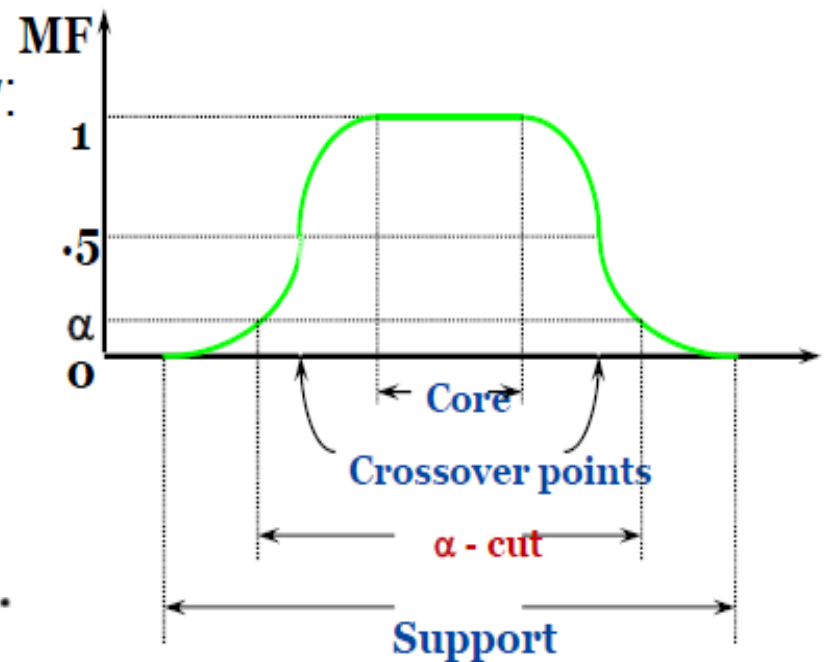
α -cut and strong α -cut

- α -cut set:
- is a crisp set defined by:

$$A_\alpha = \{x | \mu_A(x) \geq \alpha\}.$$

- Strong α -cut set:

$$A'_\alpha = \{x | \mu_A(x) > \alpha\}.$$



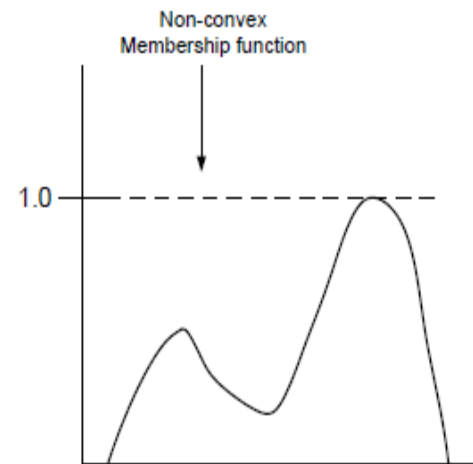
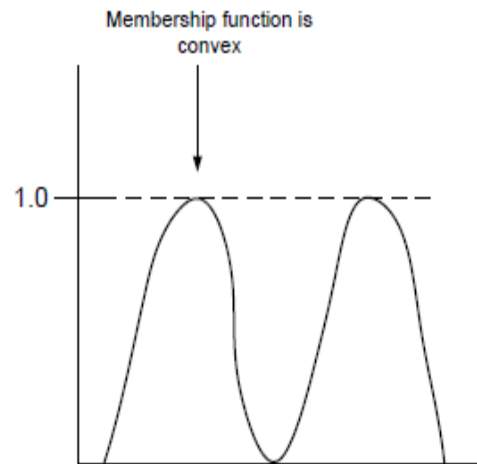
Convexity of Fuzzy Sets

Convexity : A fuzzy set A is convex if and only if for any x_1 and $x_2 \in X$ and any $\lambda \in [0, 1]$

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

Note :

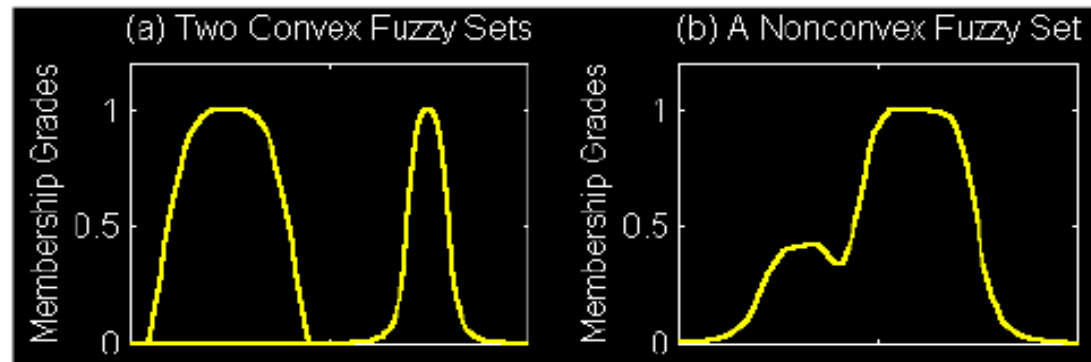
- A is convex if all its α - level sets are convex.
- Convexity (A_α) $\implies A_\alpha$ is composed of a single line segment only.



Convexity of Fuzzy Sets

- A fuzzy set A is convex if for any λ in $[0, 1]$,
$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

Alternatively, A is convex if all its α -cuts are convex.



Bandwidth

Bandwidth :

For a normal and convex fuzzy set, the bandwidth (or width) is defined as the distance the two unique crossover points:

$$\text{Bandwidth}(A) = |x_1 - x_2|$$

where $\mu_A(x_1) = \mu_A(x_2) = 0.5$

Bandwidth

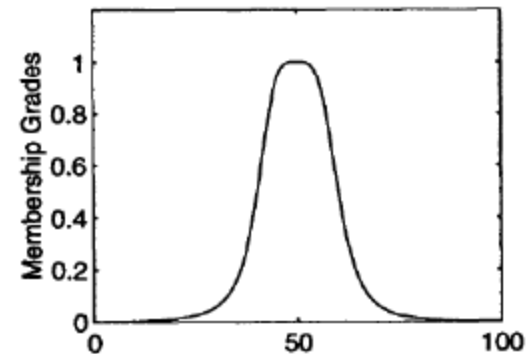
- **Bandwidth of normal and convex fuzzy sets:**
- The bandwidth or simply width of a fuzzy set is defined as the distance between its unique (why unique?) crossover points

$$\text{width}(A) = |x_2 - x_1|,$$

- where $\mu_A(x_1) = \mu_A(x_2) = 0.5$.

Fuzzy Number

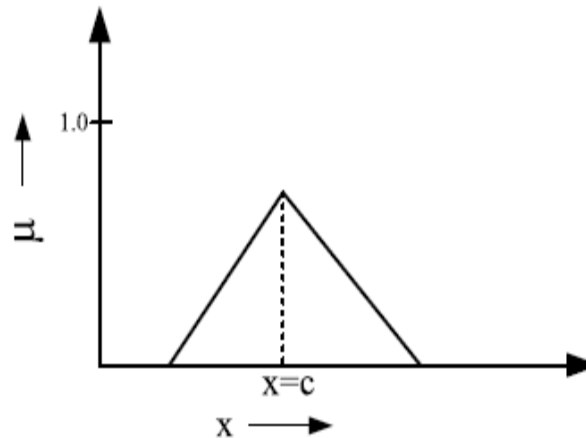
- **Fuzzy number:**
- A fuzzy number A is a fuzzy set that satisfies the condition for **normality** and **convexity**.
- Most fuzzy sets in the literature satisfy these conditions, and thus most fuzzy sets are fuzzy numbers.
- **Symmetry:**
- A fuzzy set A is symmetric if its MF is symmetric around $\mu_A(c + x) = \mu_A(c - x)$ for all $x \in X$.



Symmetry Fuzzy Set

Symmetry :

A fuzzy set A is symmetric if its membership function around a certain point $x = c$, namely $\mu_A(x + c) = \mu_A(x - c)$ for all $x \in X$.



Open left, open right and closed Fuzzy Sets

A fuzzy set A is

Open left

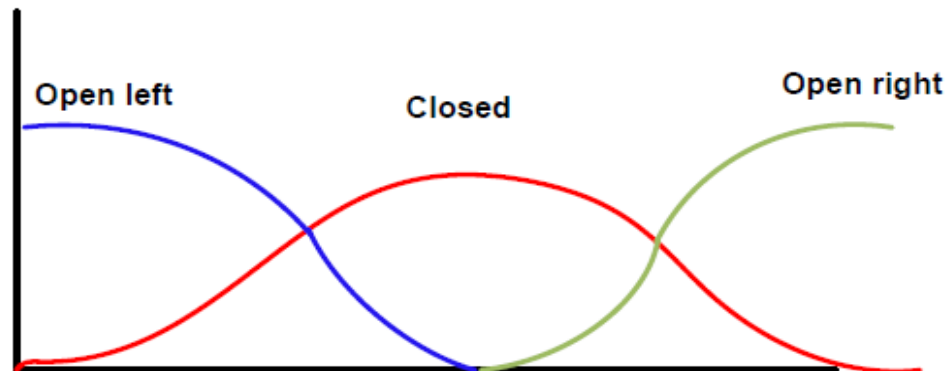
If $\lim_{x \rightarrow -\infty} \mu_A(x) = 1$ and $\lim_{x \rightarrow +\infty} \mu_A(x) = 0$

Open right:

If $\lim_{x \rightarrow -\infty} \mu_A(x) = 0$ and $\lim_{x \rightarrow +\infty} \mu_A(x) = 1$

Closed

If : $\lim_{x \rightarrow -\infty} \mu_A(x) = \lim_{x \rightarrow +\infty} \mu_A(x) = 0$



Cardinality of a Fuzzy Set

- The cardinality $|A|$ of a fuzzy set A is defined as

$$|A| = \sum_{x \in X} \mu(x)$$

- The relative cardinality of A is defined as

$$\|A\| = \frac{|A|}{|X|}$$

Fuzzy Set Theory

HOME TASK 6

- Let X be a Universal set of 'Age' represented as;
- $X = \{5, 10, 20, 30, 40, 50, 60, 70, 80, 90\}$
- 'young' and 'old' are two Fuzzy Sets defined as;
- $\text{young} = \{ (5, 1), (10, 1), (20, 0.8), (30, 0.5), (40, 0.2), (50, 0.1), (60, 0), (70, 0), (80, 0), (90, 0) \}$.
- $\text{old} = \{ (5, 0), (10, 0), (20, 0.1), (30, 0.2), (40, 0.4), (50, 0.6), (60, 0.8), (70, 1), (80, 1), (90, 1) \}$.

Fuzzy Set Theory

HOME TASK 6

Find out the following;

1. support (young)
2. support (old)
3. core (young)
4. core (old)
5. young _{0.2}
6. young' _{0.2}
7. young _{0.8}

Fuzzy Set Theory

HOME TASK 6

Find out the following;

8. $\text{young}'_{0.8}$

9. young_1

10. $\text{old}_{0.4}$

11. $\text{old}'_{0.4}$

12. $\text{old}_{0.6}$

13. old_1

14. $|\text{young}|$

15. $|\text{old}|$

MOST COMMONLY USED MEMBERSHIP FUNCTIONS IN ENGINEERING

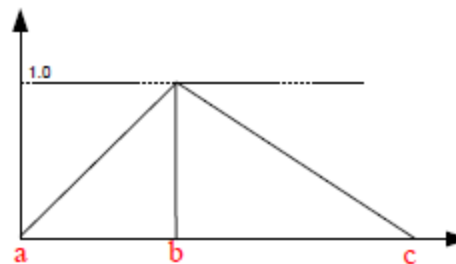
Types of Membership Functions

Triangular MFs

In the following, we try to parameterize the different MFs on a continuous universe of discourse.

Triangular MFs : A triangular MF is specified by three parameters $\{a, b, c\}$ and can be formulated as follows.

$$\text{triangle}(x; a, b, c) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } c \leq x \end{cases}$$

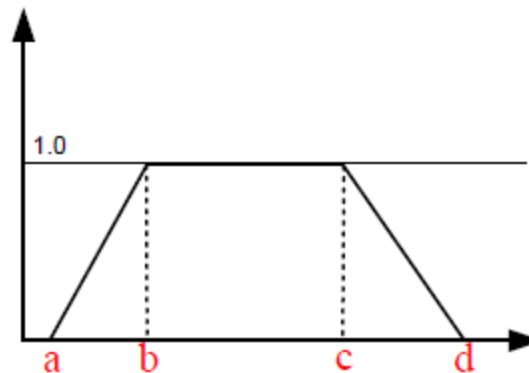


Types of Membership Functions

Trapezoidal MFs

A trapezoidal MF is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

$$\text{trapeziod}(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } d \leq x \end{cases}$$

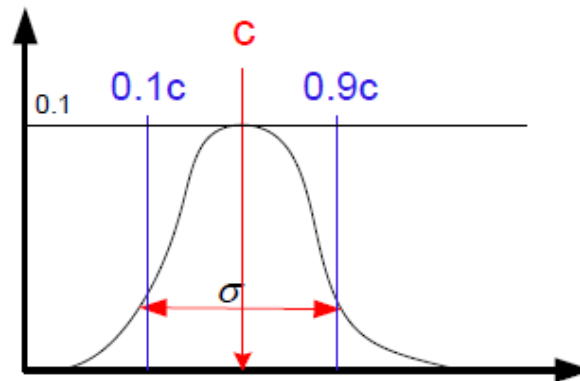


Types of Membership Functions

Gaussian MFs

A Gaussian MF is specified by two parameters $\{c, \sigma\}$ and can be defined as below:

$$\text{gaussian}(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}.$$



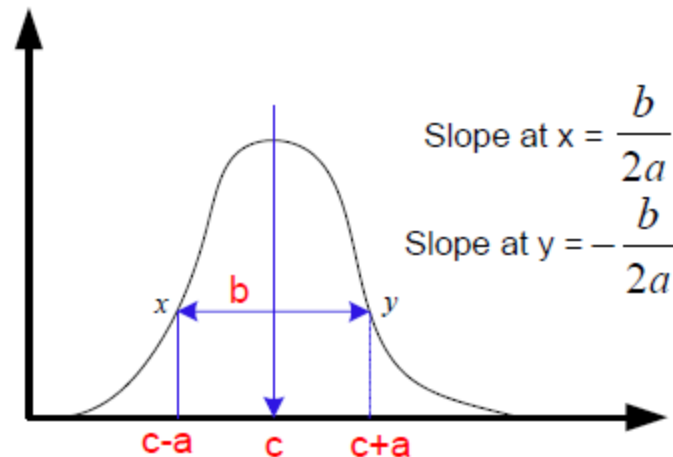
Where 'c' represents MFs Centre and σ , represents the MFs width.

Types of Membership Functions

Generalized Bell MFs

It is also called **Cauchy MF**. A generalized bell MF is specified by three parameters $\{a, b, c\}$ and is defined as:

$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left| \frac{x-c}{a} \right|^{2b}}$$

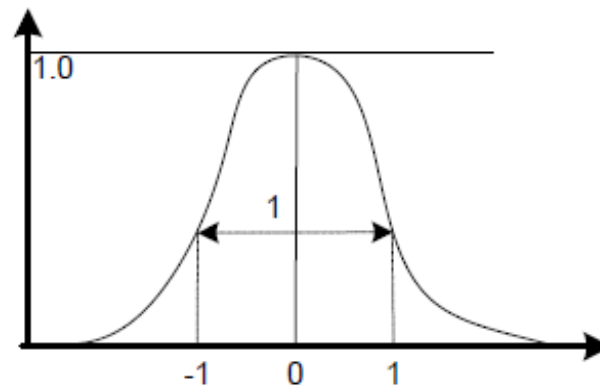


Where 'a' represents the width, 'c' is the centre and 'b' is the slope of the MFs. If 'b' is negative then the shape of the MF becomes an upside down Bell.

Types of Membership Functions

Generalized Bell MFs

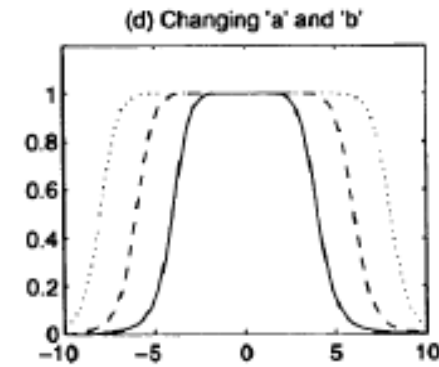
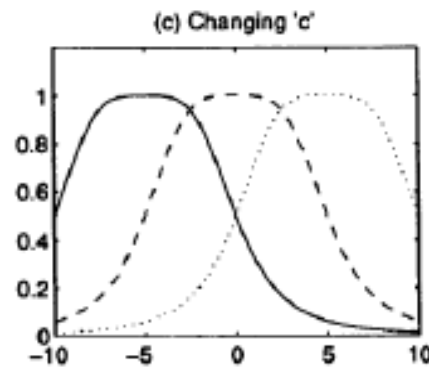
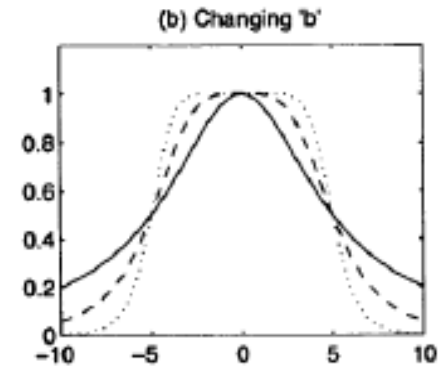
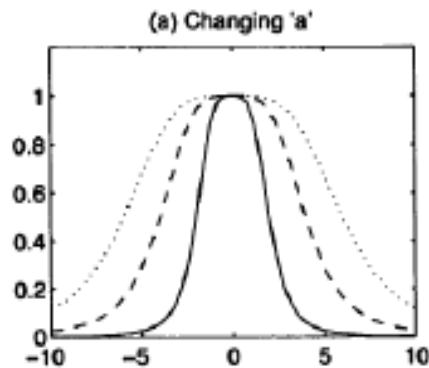
Example: $\mu(x) = \frac{1}{1+x^2}$;
 $a = b = 1$ and $c = 0$;



Types of Membership Functions

Generalized Bell MFs

- Specified by three parameters: $\{a, b, c\}$

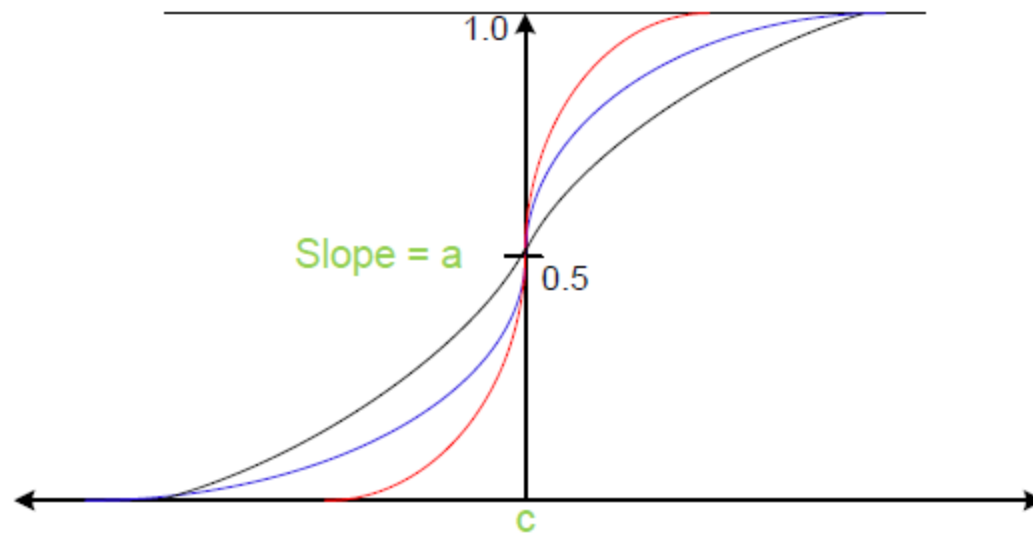


Types of Membership Functions

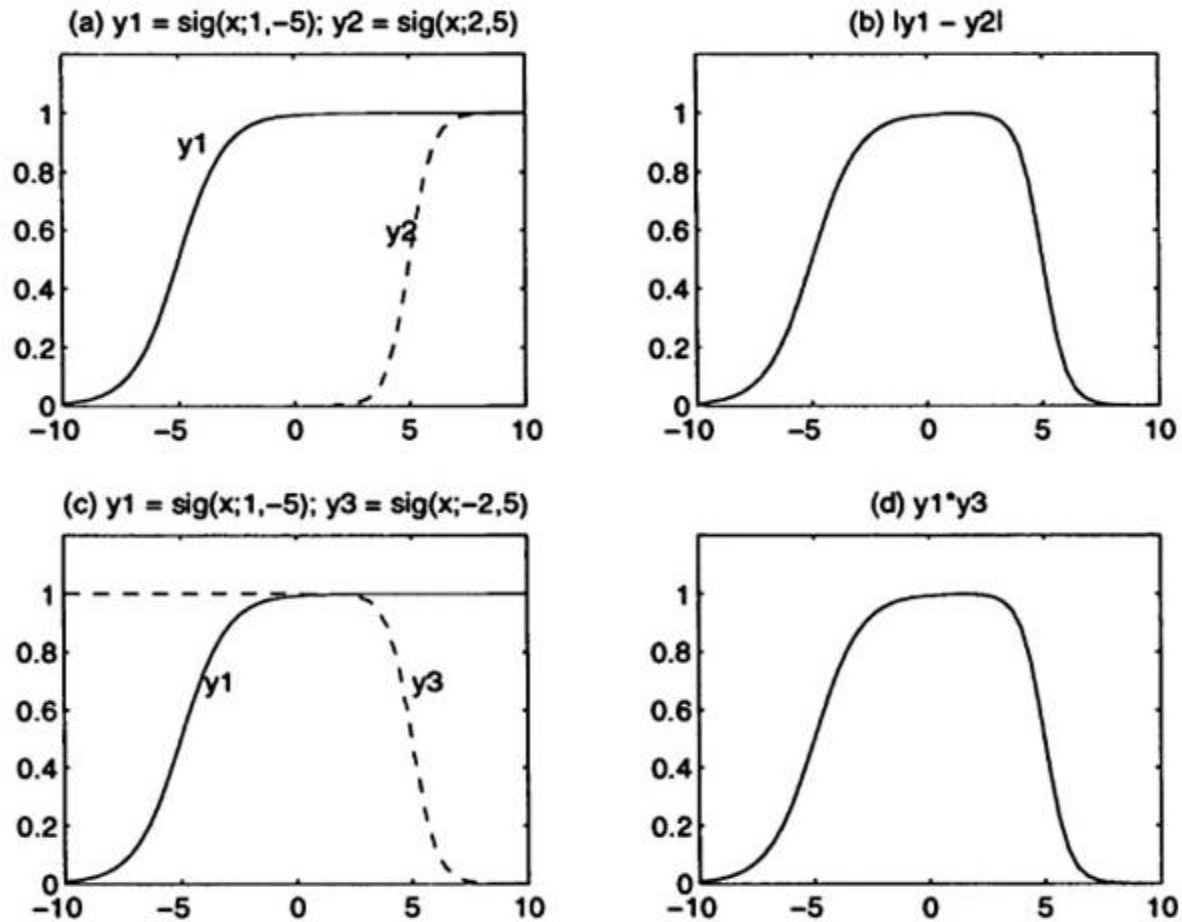
Sigmoidal MFs

Parameters: $\{a, c\}$; where c = crossover point and a = slope at c ;

$$\text{Sigmoid}(x;a,c) = \frac{1}{1 + e^{-\left[\frac{a}{x-c}\right]}}$$



Close and asymmetric MFs based on sigmoid functions



(a) Two sigmoid function y_1 and y_2 ; (b) A close MF obtained from $|y_1 - y_2|$;
(c) Two sigmoid functions y_1 and y_3 ; (d) a close MF obtained from $y_1 y_3$.

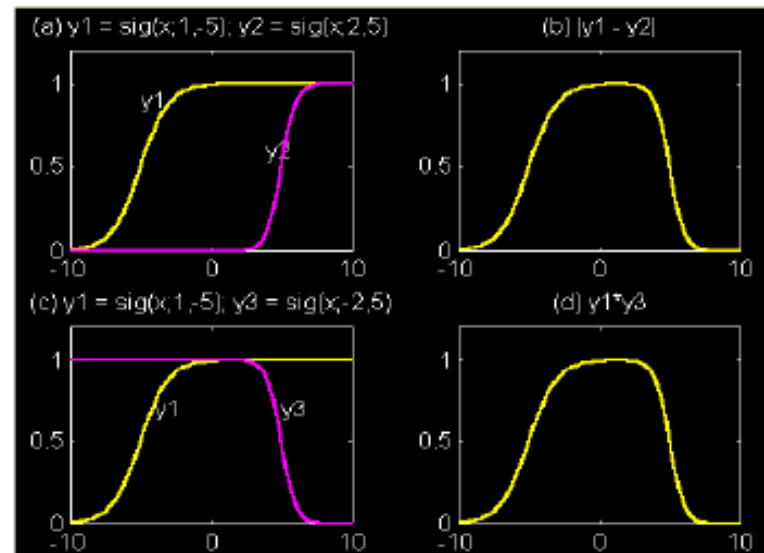
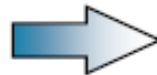
Types of Membership Functions

Sigmoidal MFs

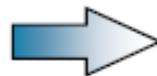
- Sigmoidal MF: $\text{sigmf}(x; a, c) = \frac{1}{1 + e^{-a(x-c)}}$

Extensions:

Absolute difference
of two sig. MFs



Product of two sig.
MFs



Types of Membership Functions

